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Plane Regions Determined by Complex Moments

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1. INTRODUCTION

Let B designate a bounded region assumed to lie in the complex $z = x + iy$ plane. The complex numbers

$$\tau_n = \iint_B z^n \, dx \, dy, \qquad n = 0, 1, 2,..., \tag{1.1}$$

will be called the *complex moments* of the region B. To what extent is B determined uniquely by the numbers τ_n or by a subset of them? If there is no uniqueness, how might the family $\mathscr{B}$, from which competing regions B have been extracted, be restricted so that uniqueness results? This is obviously a question about the completeness (in some sense) of the complex powers z^n, $n = 0, 1,...$.

In dealing with two dimensional regions, one normally deals with the real moments

$$\mu_{m,n}(B) = \mu_{m,n} = \iint_B x^m y^n \, dx \, dy, \qquad m, n = 0, 1,.... \tag{1.2}$$

Introducing the numbers $\tau_{m,n}$ by means of

$$\tau_{m,n} = \iint_B z^m \bar{z}^n \, dx \, dy, \qquad m, n = 0, 1,..., \bar{z} = x - iy, \tag{1.3}$$

one has

$$\tau_{m,n} = \sum_{\substack{j=0\\k=0}}^{m,n} (i)^{m-j} (i)^{n-k} \binom{m}{j} \binom{n}{k} \mu_{j+k,m+n-j-k} \qquad (i = (-1)^{1/2}) \tag{1.4}$$

or, in the reverse direction,

$$\mu_{m,n} = (-i)^n \, 2^{-m-n} \sum_{\substack{j=0\\k=0}}^{m,n} \binom{m}{j} \binom{n}{k} \tau_{j+k,m+n-j-k} \, . \tag{1.5}$$

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Thus, given all the moments $\mu_{m,n}$, we may determine all the moments $\tau_{m,n}$ and vice versa. The matrix

$$M = (\tau_{m,n}) \tag{1.6}$$

cannot be arbitrary. It is, for example, Hermitian ($M = M^*$), so that only the upper triangle is necessary to reconstruct all the moments $\tau_{m,n}$. Then, again, its elements must satisfy the Schwarz Inequalities

$$|\tau_{m,n}|^2 \leqslant \tau_{m,m}\tau_{n,n}. \tag{1.7}$$

With respect to the moments (1.2) or (1.3), the uniqueness situation is governed by the following typical theorems that are based upon the real powers $x^m y^n$, $m, n = 0, 1, 2, \ldots$.

THEOREM. *Let $\mathscr{B}$ designate the family of plane sets which are measurable and which lie in a fixed disc C. Let $L(C)$ designate the class of functions that are defined on C and are integrable. Let $f_n(x, y)$, $n = 1, 2, \ldots, \in L(C)$. Given B and $D \in \mathscr{B}$. A necessary and sufficient condition that*

$$\iint_B f_n(x, y)\, dx\, dy = \iint_D f_n(x, y)\, dx\, dy, \qquad n = 1, 2, \ldots, \tag{1.8}$$

imply $B = D$ a.e. is that the sequence $f_n(x, y)$ be complete in $L(C)$.

THEOREM. *Let B and D be bounded open sets which possess exterior points in the neighborhood of any boundary point. Then,*

$$\iint_B x^m y^n\, dx\, dy = \iint_D x^m y^n\, dx\, dy, \qquad m, n = 0, 1, \ldots, \tag{1.9}$$

implies $B = D$.

For proofs of these statements see, e.g., [3 pp. 198–200].

Now, the basic question we have raised here is what can be said on the basis of knowledge only of the numbers

$$\tau_n = \tau_{n,0} = \iint_B z^n\, dx\, dy, \qquad n = 0, 1, \ldots. \tag{1.10}$$

We cannot obtain all the values (1.2) from a knowledge only of (1.10)—at least, not through (1.5). (For example, we cannot express $x^2 + y^2 = z\bar{z}$ as a polynomial in $1, z, z^2$.) We are dealing with a subset of the matrix M, and hence with a problem of Müntz type.

In the present paper, we make the assumption that B is restricted to be one of the simplest of elementary figures, namely, a triangle. We shall prove that a triangle T is uniquely determined by its first four complex moments.

2. Complex Moments of a Triangle

For three arbitrary complex numbers z_1, z_2, z_3, introduce the three elementary symmetric functions

$$\begin{aligned} \text{sum} = s &= z_1 + z_2 + z_3, \\ t &= z_1 z_2 + z_2 z_3 + z_3 z_1. \\ \text{product} = p &= z_1 z_2 z_3. \end{aligned} \tag{2.1}$$

Furthermore, for $n =$ integer, set

$$G_n = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1{}^n & z_2{}^n & z_3{}^n \end{vmatrix}. \tag{2.2}$$

Note that G_2 is the Vandermonde of z_1, z_2, z_3 and

$$G_2 = (z_3 - z_1)(z_3 - z_2)(z_2 - z_1).$$

Lemma 1.

$$\frac{G_3}{G_2} = s, \qquad \frac{G_4}{G_2} = s^2 - t, \qquad \frac{G_5}{G_2} = s^3 - 2st + 7p. \tag{2.3}$$

Proof. Computation.

Note that G_n/G_2 is a symmetric polynomial in z_1, z_2, z_3, so that by the fundamental theorem of such polynomials, it must be a polynomial in s, t, and p. Lemma 1 gives the explicit representation for $n = 3, 4, 5$.

Let T designate a nondegenerate triangle whose vertices in any order are z_1, z_2, z_3. Let $A = A(T)$ designate the (positive) area of T.

Lemma 2. *Let $f(x)$ be any analytic function which is regular in T and continuous in the closure of T. Then*

$$\frac{1}{2A} \iint_T f''(z)\, dx\, dy = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ f(z_1) & f(z_2) & f(z_3) \end{vmatrix} \div G_2. \tag{2.4}$$

Proof. See [2, p. 128] and also the references cited there. We have called (2.4) the *Motzkin–Schoenberg–Grunsky formula.*

Corollary 1. *For integer $n \geqslant 2$,*

$$(n(n-1)/2A) \iint_T z^{n-2}\, dx\, dy = G_n/G_2. \tag{2.5}$$

COROLLARY 2.

$$\tau_1 = \iint_T z\,dx\,dy = \frac{A}{3}\frac{G_3}{G_2} = \frac{A}{3}s,$$
$$\tau_2 = \iint_T z^2\,dx\,dy = \frac{A}{6}\frac{G_4}{G_2} = \frac{A}{6}(s^2 - t), \tag{2.6}$$
$$\tau_3 = \iint_T z^3\,dx\,dy = \frac{A}{10}\frac{G_5}{G_2} = \frac{A}{10}(s^3 - 2st + 7p).$$

Proof. Corollary 1 and Lemma 1.

COROLLARY 3.

$$\begin{aligned} s &= 3\tau_1 \div \tau_0, \\ t &= (9\tau_1^2 - 6\tau_0\tau_2) \div \tau_0^2, \\ p &= (10\tau_0^2\tau_3 + 27\tau_1^3 - 36\tau_0\tau_1\tau_2) \div \tau_0^3. \end{aligned} \tag{2.7}$$

Proof. $A = \tau_0$. Now solve (2.5) successively.

THEOREM. *A triangle T is determined uniquely by its first four complex moments $\tau_k = \iint_T z^k\,dx\,dy$, $k = 0, 1, 2, 3$.*

Proof. The first four moments determine s, t, p uniquely through (2.7). Now since $\phi(z) = (z - z_1)(z - z_2)(z - z_3) = z^3 - sz^2 + tz - p$, the polynomial $\phi(z)$ is determined uniquely. Hence also the roots z_1, z_2, z_3, up to a permutation.

COROLLARY. *A triangle in the x, y plane is determined uniquely by the seven real numbers*

$$A(T) = \iint_T dx\,dy, \qquad \iint_T x\,dx\,dy, \qquad \iint_T y\,dx\,dy, \qquad \iint_T (x^2 - y^2)\,dx\,dy,$$
$$\iint_T xy\,dx\,dy, \qquad \iint_T (x^3 - 3xy^2)\,dx\,dy, \qquad \iint_T (3x^2y - y^3)\,dx\,dy.$$

Proof. Split τ_k, $k = 0, 1, 2, 3$ into real and imaginary parts.

It should be observed that the identities (2.7) coupled with $\phi(z)$ provide an algorithm for constructing T from given values τ_0, τ_1, τ_2, τ_3. Note further that four arbitrary numbers are not necessarily the first four moments of a T. *Example*: $\tau_0 = A > 0$, $\tau_1 = 0$, $\tau_2 = \tau_3 = 0$.

3. SOME COUNTEREXAMPLES

(a) *Sets of measure* 0. If B and D differ by a plane set of measure 0, then, of course, $\mu_{m,n}(B) = \mu_{m,n}(D)$ and $\tau_{m,n}(B) = \tau_{m,n}(D)$. This is mentioned because B and D may be essentially different as far as conformal mapping is concerned. For example: $B =$ unit disc., $D =$ unit disc with $0 \leqslant x \leqslant 1$ removed.

(b) *Connectivity.* Let B and D be two distinct annuli centered at $z = 0$ and such that area$(B) =$ area(D). Then, as an easy computation shows, $\tau_n(B) = \tau_n(D)$, $n = 0, 1, 2,\ldots$.

(c) *Algebraic dependence.* It might be thought, in view of the above theorem, that any four numbers $\tau_{m,n}$ will determine a triangle uniquely. But this is not so. For example, the four numbers $\tau_{0,0}$, $\tau_{1,0}$, $\tau_{2,0}$, $\tau_{1,1}$ do not serve to determine T uniquely.

To show this, let T_1 and T_2 be any two equilateral triangles whose area is a fixed constant A and whose center of gravity is at $z = 0$. Therefore $\tau_{0,0}(T_1) = \tau_{0,0}(T_2) = A$. If $z_{\text{c.g.}}$ designates the center of gravity of a triangle T, then $z_{\text{c.g.}} = (1/A) \iint z\, dx\, dy = s/3$. Therefore, $\tau_{1,0}(T_1) = \tau_{1,0}(T_2) = 0$. If T is equilateral, then its vertices can be represented as $z_1 = z^*$, $z_2 = z^*\omega$, $z_3 = z^*\omega^2$ with $\omega^3 = 1$ and $|z^*| = \sigma$. Therefore $t = z_1z_2 + z_2z_3 + z_3z_1 = z^{*2}(\omega + 1 + \omega^2) = 0$. Thus, $t(T_1) = t(T_2) = 0$ so that by (2.6), $\tau_{2,0}(T_1) = \tau_{2,0}(T_2)$.

Finally, for any triangle T,

$$\iint_T z\bar{z}\, dx\, dy = (A/12)(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2). \quad (3.1)$$

(See [1, p. 575].) Therefore,

$$\begin{aligned}\tau_{1,1}(T_1) &= \iint_{T_1} z\bar{z}\, dx\, dy = (A/12)(|z_1|^2 + |z_2|^2 + |z_3|^2) = A\sigma^2/4 \\ &= \tau_{1,1}(T_2).\end{aligned}$$

What is involved here, of course, are moment *invariants*. For example, if $m + p = n + q$ then the quantities $\tau_{m,n}\tau_{p,q}$ are invariant under rotations about the origin. The question of invariance under certain groups is of importance in pattern recognition.

In a paper which follows, we plan to apply the ideas here to the problem of the approximation of one region by another.

References

1. P. J. Davis, Triangle formulas in the complex plane, *Math. Comp.* **18** (1964), 569–577.
2. P. J. Davis, "The Schwarz Function and Its Applications," Carus Monograph No. 17, *Math. Assoc. Amer.*, Washington, D.C., 1974.

3. P. J. Davis and H. O. Pollak, On the analytic continuation of mapping functions, *Trans. Amer. Math. Soc.* **87** (1958), 198–225.
4. J. A. Tamarkin and J. D. Shohat, "The Problem of Moments," Mathematical Surveys, No. 1, *Amer. Math. Soc.*, New York, 1943.
5. M-K. Hu, Visual pattern recognition by moment invariants, *IRE Trans. Information Theory* **IT-8** (1962), 179–187.
6. F. L. Alt, Digital pattern recognition by moments, *J. Assoc. Comput. Mech.* **9** (1962), 240–258.